Performance Based Learning and Assessment Task
Triangles in Parallelograms

I. ASSESSSMENT TASK OVERVIEW & PURPOSE:
In this task, students will discover and prove the relationship between the triangles formed from parallelograms.

II. UNIT AUTHOR:
Ashley Swandby, James River High School, Botetourt County Public Schools

III. COURSE:
Geometry

IV. CONTENT STRAND:
Geometry

V. OBJECTIVES:
Students will prove the relationship between triangles formed when constructing possible vertices for a parallelogram given three non-collinear points. Students will also classify/prove the quadrilateral formed when connecting midpoints of sides of parallelograms.

VI. REFERENCE/RESOURCE MATERIALS:
Students may choose to construct figures using dynamic geometry software, but the activities also work with straightedges and paper/pencil.

VII. PRIMARY ASSESSMENT STRATEGIES:
Students will communicate reasoning/justification of the following:

-existence of a fourth point given any three non-collinear points to create a parallelogram
-existence of three such points such that given three non-collinear points, a parallelogram can be formed
-the four triangles constructed from three given points and three additional points to construct parallelograms create four congruent triangles
-the quadrilateral constructed by connecting midpoints of sides of parallelograms create another parallelogram
Students should use figures, dynamic geometry software, or constructed sketches to visualize properties of the triangles and parallelograms as well as provide formal justification of the results

VIII. EVALUATION CRITERIA:
For each question, students should provide a sketch and a formal justification of their conclusions. A possible justification is provided. Students should make explicit justification that the three points on each side of the larger triangle in Question 2 are in fact collinear. Students should be able to construct all three of the possible vertices to create parallelograms from any three non-collinear points before they attempt to prove the resulting triangles congruent.

IX. INSTRUCTIONAL TIME:
30 minutes

Constructing Parallelograms and Triangles

Strand
Geometry

Mathematical Objective(s)
Students will provide a formal justification of the relationship between the triangles formed when constructing the possible vertices to create a parallelogram from three non-collinear points. Students will also provide a formal justification for the type of quadrilateral created when connecting midpoints of sides of parallelograms.

Related SOL

G.9: The student will verify characteristics of quadrilaterals and use properties of quadrilaterals to solve real-world problems.
G.6: The student, given information in the form of a figure or statement, will prove two triangles are congruent, using algebraic and coordinate methods as well as deductive proofs.
G. 7: The student, given information in the form of a figure or statement, will prove two triangles are similar, using algebraic and coordinate methods as well as deductive proofs.
G.2 : The student will use the relationships between angles formed by two lines cut by a transversal to
b) verify the parallelism, using algebraic and coordinate methods as well as deductive proofs; and
c) solve real-world problems involving angles formed when parallel lines are cut by a transversal.

NCTM Standards The student will:

- **Geometry:** analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships
  - **(9-12)** explore relationships (including congruence and similarity) among classes of two- and three-dimensional objects, make and test conjectures about them, and solve problems involving them
  - **(9-12)** establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others
- **Geometry:** specify locations and describe spatial relationships using coordinate geometry and other representational systems
  - **(9-12)** investigate conjectures and solve problems involving two- and three-dimensional objects represented with Cartesian coordinates

- **Geometry:** use visualization, spatial reasoning, and geometric modeling to solve problems
    - **(9-12)** draw and construct representations of two- and three-dimensional geometric objects using a variety of tools
- **Reasoning and Proof:** make and investigate mathematical conjectures
- **Reasoning and Proof:** develop and evaluate mathematical arguments and proofs
- **Communication:** organize and consolidate their mathematical thinking through communication; communicate their mathematical thinking coherently and clearly to peers, teachers, and others

Additional Objectives for Student Learning (include if relevant; may not be math-related):

Materials/Resources

Dynamic Geometry Software, if using for constructions

Straightedge and compass, if using for constructions

Assumption of Prior Knowledge

- Students should be able to prove relationships for angles in parallel lines
- Students should be able to prove congruent and/or similar triangle relationships
- Students should be able to identify properties of parallelograms and how to prove a quadrilateral is a parallelogram
- Students might have trouble identifying the three possible vertices that create parallelograms from any three non-collinear points. Students may also have trouble in the proof identifying why the three points on the sides of the larger triangle are in fact collinear. Students should be prompted to justify this linearity relationship if necessary.
- Before this task, students should also have considered properties of parallelograms in the coordinate plane

Introduction: Setting Up the Mathematical Task

In this task, the student will prove the relationship between the vertices and midpoints of parallelograms and triangles. The student will justify and prove the relationships between triangles and quadrilaterals formed when constructing various line segments. The student should create a visual representation (either with dynamic geometry software or with paper/pencil) to explore the relationships and then must provide a formal justification for the conclusions.

The questions presented in the task is as follows: When three non-collinear points are on a plane, where can you draw a fourth point in order to guarantee the four points will create a parallelogram? Is it always possible to construct a parallelogram from three non-collinear points? Can there be more than one point that creates a parallelogram? What conclusions can you make about the triangles formed from the points that create all possible parallelograms?

For this task, the student should apply your knowledge of parallelograms, parallel lines and angles formed by parallel lines cut by a transversal, and properties of congruent or similar triangles in order to justify reasoning. For each question, the student should provide a figure and a written justification of the thinking involved.

Teacher Note: (Students should complete this task in pairs in order to promote communication and reasoning/proof. The teacher should be observing the discussions and justifications and providing feedback to ensure the students are giving complete justifications of their conclusions.) Students may want to use dynamic geometry software to assist in the constructions, but ensure that the students are using the construction methods and geometrical properties to provide justification instead of relying solely on the observations from the software (Ask: why is that always true? or What allows you to make that conclusion?) It may be helpful to give the questions one at a time and have students present their models to the class after Question 1 in order for students to discover that there are three possible points that can form a parallelogram. Encourage students to justify their constructions and critique each other's justification. As students provide justification, ensure that the conclusions reached are justified mathematically, not simply based on observation (for example, why must three points be on the same line? Will any three points always be on the same line? Why must certain angles/sides be congruent in the construction?)

Student Exploration

1. Consider three non-collinear points on the plane. Label them A, B, and C. From these points, where can you draw a fourth point, D, such that a parallelogram is formed? How do you know that this point exists?

2. When given three non-collinear points, there are actually three different points that can create a possible parallelogram with the given points. Where are these three points? Draw a figure that includes these three points. Label the given points A, B, and C, and the new points D, E, and F. From your drawing, you will notice four triangles. What is the relationship between these four triangles? Provide a formal justification for your conclusion.

Monitoring Student Responses

- Students should provide a figure and a written justification for each question posed. Students should discuss their thinking with partners, and the teacher should listen for students' misconceptions in order to guide their thinking and ensure proper justification. If students struggle to identify the relationship, prompt students to make an observation by measuring sides and angles and then try to explain why they think the observation must be true.

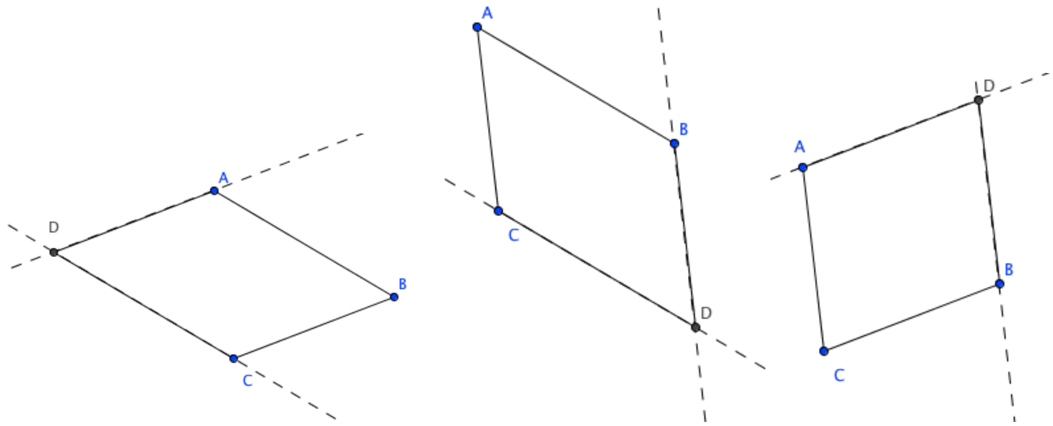
- After students have prepared written justifications, have student groups present their proofs to the class. Encourage students to discuss their thinking throughout the task and why they chose to approach the task in their chosen way. How else could the conclusions be proven?

Assessment List and Benchmarks

- Extension Questions
  - **Questions**
    - If three vertices of a parallelogram are located at (0, 0), (3, 4) and (-2, 3), where could the fourth vertex be located? Is there more than one possible location? How do you know that must be the location?
  - **Journal/writing prompts**
    - What initial conditions would ensure that the four triangles constructed are isosceles? Equilateral?
    - Why is it possible to construct three different parallelograms from any three non-collinear points?
- Students should be assessed on the following: (5 point scale)
  - accurate model provided (1 point)
  - complete justifications for each question (are all statements supported with geometrical reasoning?)—4 points for complete justification, 3 points if minor assumption made without justification, 2 points for substantial missing justification, and 1 point for complete lack of justification but an accurate conclusion. 0 points for incorrect conclusions.

- Possible solutions:

Question 1: There are three possible parallelograms that can be constructed from the three non-collinear points.

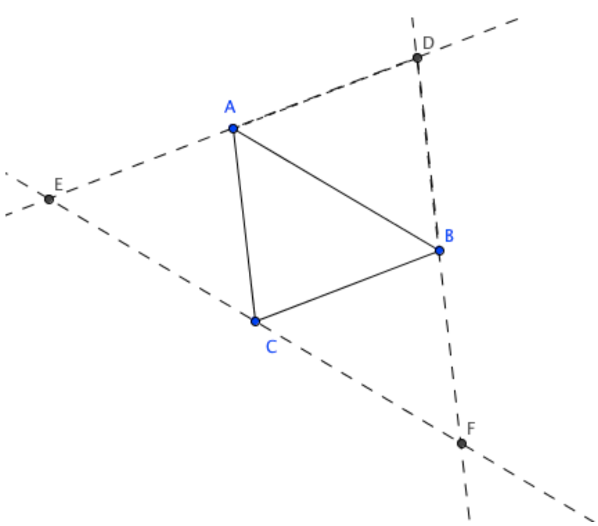


Possible justifications include:

A) Construct a line segment through two points, draw a line segment parallel and congruent to the line through the other point. The fourth point occurs at the endpoint of the parallel and congruent line segment.

B) Construct a line segment through two points, and construct a line segment through another pair of points. From each line segment, construct a line segment parallel to the line through the other point. The fourth point occurs at the intersection of the line segments.

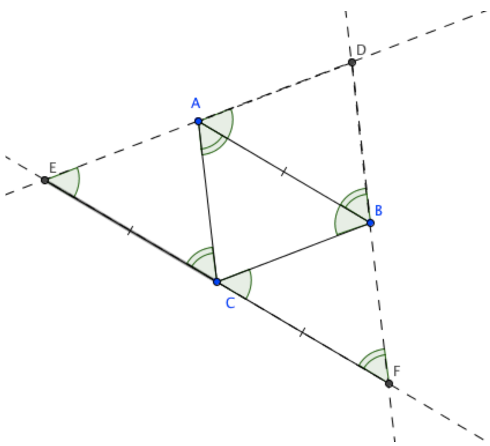
Question 2:



The four smaller triangles ($\triangle ABD, \triangle ACE, \triangle BCF, \triangle ABC$ are congruent and they are all similar to the larger triangle, $\triangle DEF$

There are multiple approaches, and the corresponding sides and angles will vary. Ensure that students use an appropriate congruence justification and provide adequate justification. A possible justification approach could include this reasoning:

- Use ASA for the four triangles. For each of the pairs of parallel lines, congruent angles are formed by corresponding angles for each of the three outside triangles and alternate interior angles for the interior triangle. For each parallelogram formed, we conclude that opposite sides are congruent and prove that the four triangles are congruent by ASA.



- Use SSS by concluding corresponding sides of the parallelograms are congruent and use the transitive property to conclude congruence for all four triangles.